Finding a Common Set of Weights for Ranking Decision-Making Units in Data Envelopment Analysis

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***Abstract*—Data envelopment analysis (DEA) has been a popular method of measuring production performance of a group of decision-making units (DMUs) utilizing same types of inputs to produce outputs. DEA has been applied to many studies including studies on the performance of hospitals, banks, airlines, and schools. However, despite of its popularity, DEA still suffers from some inadequacies. For instance, it sometimes may overestimate efficiencies of some DMUs by applying weight sets with extreme values. Furthermore, it lacks discriminating power among efficient DMUs, since all efficient DMUs have the same efficiency scores. In this paper, we propose a methodology which attempts to identify DMUs which efficiencies are possibly overstated by DEA and provides a common set of input and output weights for ranking DMUs. A common set of weights sometimes is useful when studying tradeoffs between different inputs and outputs in DEA especially when different inputs can be traded and obtaining a full rank of DMUs.**

***Index Terms*—Common set of weights, cross-efficiency score, data envelopment analysis, performance measure.**

I. Introduction

Charnes, Cooper, and Rhodes [1] introduced data envelopment analysis (DEA) as a method to measure relative production performance among a group of decision-making units (DMUs). Since its introduction, many studies have applied DEA to measure production performance of a variety of DMUs. Nowadays, DEA becomes a very popular and important method in performance measure. Suppose there are n decision making units, DMU_j ($j = 1,2,\ldots,n$), each of which has m inputs x_{ij} ($i = 1,\ldots, m$) and s outputs y_{rj} ($r = 1,\ldots, s$). The maximum relative efficiency of a decision-making unit, DMU_o can be obtained from the following linear programming (LP) model [1], usually known as CCR model:

$$\text{Max} \sum_{r=1}^{s} u_r y_{ro} \tag{1}$$

$$\text{S.T.} \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \;\; j = 1,\ldots,n \tag{2}$$

$$\sum_{r=1}^{s} v_i x_{io} = 1,$$

$$u_r, v_i \ge 0, r = 1,\ldots, s; i = 1,\ldots, m \tag{3}$$

where u_r ($r = 1,\ldots, s$) and v_i ($i = 1,\ldots, m$) are factor weights for output r and input i, respectively. DMU_o is efficient if, $\sum_{r=1}^{s} u_r y_{ro} = 1$, otherwise is inefficient. Another version of the CCR model requires u_r, $v_i \ge \varepsilon$, where $r = 1,\ldots, s$; $i = 1,\ldots, m$; and ε is a very small positive number.

Recently, DEA models using common set of weights have been studied by some researchers. Lam [2] uses common set of weights to determine the most efficient DMU in DEA. Ramezani-Tarkhorani, Khodabakhshi, Mehrabian and Nuri-Bahmani [3] rank DMUs using common set of weights in DEA. Omrani [4] applies common weights data envelopment analysis with uncertain data. Chiang, Hwang and Liu [5] use a separation vector to determine a common set of weights. Lam and Bai [6] minimize the input and output weights from their means using common set of weights. Wang, Luo and Lan [7] use common weights for fully ranking DMUs by regression analysis. Jahanshahloo, Hosseinzadeh Lotfi, Khanmohammadi, Kazemimahesh and Rezaie [8] rank DMUs by positive ideal DMU with common weights. Lam [9] determines appropriate weight set to compute cross efficiency in DEA.

One of the drawbacks in DEA is that all efficient DMUs have the same efficiency score; this makes it very difficult to differentiate between more efficient units to less efficient units among the efficient group. Another common weakness of DEA is that extremely diverse or unusual values of some input or output weights which are difficult to explain in the context of a rational economy might be obtained for the target unit. These weights could lead to overstating the efficiencies of some DMUs. This can be explained by the excess flexibility in choosing weights in DEA. Some DMUs become efficient only because extreme values of weight sets are used to maximize their efficiencies. In this paper, we propose a two-step methodology finding a common set of weights. Step one identifies DMUs which efficiencies are possibly overstated by DEA. Then based on the results from step one, step two determines a set of input and output weights to rank DMUs.

II. Linear Programming and Discriminant Analysis

Let A be an ($n\times k$) matrix representing k attribute scores of a known sample of n observations from two groups, G_1 and G_2. Hence, a_{ij} is the value of the i^{th} attribute for the j^{th} observation in the sample. Freed and Glover [10] proposed a linear programming model, minimizing the sum of deviations (MSD) in discriminant analysis as follows:

$$\text{Max} \sum_{j=1}^{n} d_j \tag{4}$$

Manuscript received July 6, 2016; revised September 13, 2016.

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doi: 10.18178/joebm.2016.4.9.448

$$\text{S.T.} \sum_{i=1}^{k} a_{ij} w_i + d_j \geq h, \qquad \forall j \in G_1, \tag{5}$$

$$\sum_{i=1}^{k} a_{ij} w_i - d_j \leq h, \qquad \forall j \in G_2, \tag{6}$$

where h and w_i, for i=1,…, k, are unrestricted in sign and $d_j \geq 0$, for j=1,…, n.

A normalization constraint is needed in MSD to avoid trivial solutions (i.e. solutions where all variables equal to zero). Examples of normalization constraints can be found in [11], [12]. Other forms of LP models in discriminant analysis can be found in [13], [14].

III. A Two-Step Method

We propose to use a method utilizing both cross-efficiency in DEA and linear programming in discriminant analysis, to identify less efficient DMUs in the efficient set and to provide a common set of weights for performance rankings of DMUs. The proposed method requires two steps:

A. Step One

DEA is used to determine the efficiency score and the weight set for each DMU. Then based on the n weight sets obtained from DEA, calculate the average cross-efficiency score for each DMU. The average cross-efficiency score of a DMU is computed by taking the average of n efficiency scores which are calculated by the n weight sets obtained from DEA. Let λ be the highest average cross-efficiency score among all inefficient DMUs. We then divide DMUs into two classes. Class 1 (C_1) contains efficient DMUs where their average cross-efficiency scores are also greater than λ. Class 2 (C_2) contains inefficient DMUs and also efficient DMUs which average cross-efficiency scores are less than or equal to λ. In short, DMUs in C_2 are less efficient DMUs since either they are inefficient or they have relatively low average cross-efficiency scores. The intuition behind this is that since DEA sometimes may overestimate efficiencies of some DMUs by using extremely diverse or unusual values of weights, as a result, cross-efficiency scores may be used to detect efficiencies of efficient DMUs by comparing their average cross-efficiency scores with λ. If the average cross-efficiency score of an efficient DMU is less than λ, then we have reasons to belief that those DMUs are not as efficient as other efficient DMUs and its efficiencies may be overstated.

B. Step Two

We solve the following MILP model:

$$\text{Min} \sum_{r=1}^{s} z_j \tag{7}$$

$$\text{S.T.} \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + M z_j \geq 0, \;\; j \in C_1 \tag{8}$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - M z_j \leq -\varepsilon \,, \;\; j \in C_2 \tag{9}$$

$$\frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{m} v_i x_{ij} = 1, \tag{10}$$

where $z_j \in \{0,1\}$, ε is a very small positive number, M is a large positive number; u_r, $v_i \geq 0$, r = 1, . . . , s; I = I, . . . , m.

DMUs in C_1 are expected to be more efficient than those in C_2. Constraints in (8) enforce the efficiency scores of DMUs in C_1 to be greater than or equal to one. If the efficiency score is less than one, then value of z_j is forced to one and receives a unit penalty in the objective value. On the contrary, DMUs in C_2 are expected to be less efficient than DMUs in C_1, as a result, if the weighted sum of output is not less than the weighted sum of input by a magnitude of ε, then values of z_j are forced to one by constraints in (9). The objective function of MILP minimizes the sum of z_j, or in other words, it minimizes the number of misclassifications according to the classification scheme of C_1 and C_2 obtained from Step one above. Constraint (10) is a normalization constraint to avoid the trivial solution. The obtained u_r and v_i can be used to compute efficiency scores for all DMUs. In the next section, we will examine the performance of MILP in ranking DMUs via a simulation experiment.

IV. A Simulation Experiment

In DEA, efficiency of a DMU is measured by an efficiency score, which is the sum of weighted outputs divided by the sum of weighted inputs of the DMU as shown below:

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}. \tag{11}$$

Efficient DMUs have efficiency scores equal to one, while inefficient DMUs have efficiency scores less than one.

In this simulation experiment we assume that the production function takes the following simple linear form:

$$\sum_{i=1}^{m} v_i x_{ij} = \sum_{r=1}^{s} u_r y_{rj} + \varepsilon, \tag{12}$$

where E(ε) = 0.

We generate data set with three inputs and three outputs, that is m = s = 3. All input scores and output scores are generated from a uniform distribution. Three different weight sets are used, namely, W_1 = {v_1=1; v_2=1; v_3=1; u_1=1; u_2=1; u_3=1}, W_2 = {v_1=1; v_2=4; v_3=7; u_1=1; u_2=4; u_3=7} and W_3 = {v_1=1; v_2=1; v_3=1; u_1=1; u_2=4; u_3=7}. Furthermore, we generate three different classes of DMUs, namely, Super-Efficient DMU, Efficient DMU, and Inefficient DMU. To simulate an Efficient DMU, the sum of weighted inputs equals the sum of weighted outputs as in equation (12). To simulate an Inefficient DMU, we randomly add more inputs to and subtract some amount of outputs from an Efficient DMU. As a result, for an Inefficient DMU, the sum of weighted inputs is more than the sum of weighted outputs. Similarly, to simulate a Super-Efficient DMU, we randomly subtract some inputs from and add more outputs to an efficient DMU. Hence, for a Super-Efficient DMU, the sum of weighted inputs is less than the sum of weighted outputs. In two-third of the simulated cases we generate only Efficient DMUs and Inefficient DMUs. In one-third of the simulated cases, we include Super-Efficient DMUs to emphasize the situation that in reality some efficient DMUs may perform better than other efficient DMUs, since in reality, it is very unlikely that all efficient DMUs have the same degree of efficiency.

We generate 18 different cases using two different sample sizes, three different values of weight sets, and three different mixes of efficient and inefficient DMU. The layouts of the 18 cases are summarized in Table I.

TABLE I: LIST OF 18 CASES IN THE SIMULATION EXPERIMENT

Case	Sample Size	Weight Set $\{v_1\ v_2\ v_3\}$ $\{u_1\ u_2\ u_3\}$	Mix of DMUs in the three Efficiency Classes: (A E I)*
1	60	{1 1 1} {1 1 1}	(0.2 0.2 0.6)
2	60	{1 1 1} {1 1 1}	(0.0 0.4 0.6)
3	60	{1 1 1} {1 1 1}	(0.0 0.2 0.8)
4	60	{1 4 7} {1 4 7}	(0.2 0.2 0.6)
5	60	{1 4 7} {1 4 7}	(0.0 0.4 0.6)
6	60	{1 4 7} {1 4 7}	(0.0 0.2 0.8)
7	60	{1 1 1} {1 4 7}	(0.2 0.2 0.6)
8	60	{1 1 1} {1 4 7}	(0.0 0.4 0.6)
9	60	{1 1 1} {1 4 7}	(0.0 0.2 0.8)
10	20	{1 1 1} {1 1 1}	(0.2 0.2 0.6)
11	20	{1 1 1} {1 1 1}	(0.0 0.4 0.6)
12	20	{1 1 1} {1 1 1}	(0.0 0.2 0.8)
13	20	{1 4 7} {1 4 7}	(0.2 0.2 0.6)
14	20	{1 4 7} {1 4 7}	(0.0 0.4 0.6)
15	20	{1 4 7} {1 4 7}	(0.0 0.2 0.8)
16	20	{1 1 1} {1 4 7}	(0.2 0.2 0.6)
17	20	{1 1 1} {1 4 7}	(0.0 0.4 0.6)
18	20	{1 1 1} {1 4 7}	(0.0 0.2 0.8)

*A=Super-Efficient DMU, E=Efficient DMU, I=Inefficient DMU

In the simulation experiment, we want to compare the performance of the two methods: DEA and MILP in ranking efficiencies of DMUs. The performance is measured by the correlations between efficiency scores of the method and the true efficiency scores. The true efficiency score is the simulated score. The higher the correlation, the closer is the estimated ranking to the true ranking, hence the better the performance. We perform the following paired t-test to compare the performance of ranking DMUs using DEA and MILP:

H_o: There is no difference between the mean Spearman's rho correlation coefficient of MILP with the true efficiency scores and the mean Spearman's rho correlation coefficient of DEA scores with the true efficiency scores.

H_a: The mean Spearman's rho correlation coefficient of MILP with the true efficiency scores is greater than the mean Spearman's rho correlation coefficient of DEA scores with the true efficiency scores.

The results, in terms of mean Spearman's rho correlation coefficients, standard deviations are reported in Table II. In all cases, H_o is rejected at $\alpha = 0.000001$ level.

In all cases, the mean Spearman's rho correlation coefficients of MILP with the true efficiency scores are greater than the mean Spearman's rho correlation coefficients of DEA scores with the true efficiency scores. The differences are all statistically significant. This result implies that using a common weight set, MILP can provide a more accurate ranking of DMUs than DEA. It is also observed that the larger the sample size (n=60 compares with n=20) the better is the performance of both methods.

TABLE II: MEAN* SPEARMAN'S RHO CORRELATION COEFFICIENTS AND STANDARD DEVIATIONS** BETWEEN THE TRUE EFFICIENCY SCORES AND THE EFFICIENCY SCORES OF DEA, AND BETWEEN THE TRUE EFFICIENCY SCORES AND THE EFFICIENCY SCORES OF MILP

Case	DEA efficiency scores with the true scores	MILP-efficiency scores with the true scores
1	0.6550 (0.0733)	0.8670 (0.0872)
2	0.6997 (0.0756)	0.9102 (0.0398)
3	0.5425 (0.0801)	0.9014 (0.0585)
4	0.6752 (0.0731)	0.8368 (0.0897)
5	0.7295 (0.0739)	0.9109 (0.0447)
6	0.5944 (0.0648)	0.8881 (0.0679)
7	0.6547 (0.0778)	0.8513 (0.1056)
8	0.8130 (0.0760)	0.9629 (0.0489)
9	0.7579 (0.0843)	0.9360 (0.0839)
10	0.4935 (0.1212)	0.7912 (0.1123)
11	0.5012 (0.1456)	0.7885 (0.1025)
12	0.3829 (0.1371)	0.7836 (0.1358)
13	0.5321 (0.1266)	0.7787 (0.1216)
14	0.5486 (0.1409)	0.7746 (0.1166)
15	0.4291 (0.1470)	0.7541 (0.1507)
16	0.5657 (0.1650)	0.7829 (0.1754)
17	0.6465 (0.2021)	0.8460 (0.1622)
18	0.5934 (0.1928)	0.7992 (0.1708)

*Mean of 500 trials.

**Value in parentheses is standard deviation.

V. CONCLUSION

This paper proposed a method which attempts to identify DMUs that may be overstated in terms of efficiencies by DEA and also determines a common set of weights via linear programming for the purpose of ranking efficiencies of DMUs. A common set of weights is useful when studying tradeoffs between different inputs and outputs especially when different inputs can be traded. The proposed method performs well in providing an efficiency ranking of DMUs.

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