An Inventory Model Involving Safety Factor When the Received Quantity Is Uncertain

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***Abstract*—In practice the quantity received may not match the quantity ordered due to various reasons such as rejection during inspection, human errors in counting, damage or breakage during transportation and worker's strike, etc. Under this background, we investigate a continuous review inventory model with shortage including the case where the quantity received is uncertain, in which the lead time, safety factor, lost sales rates and order processing cost are decision in variables. The objective of this paper is to minimize the total relevant cost by simultaneously optimizing the order quantity, safety factor, lost sales rates and order processing cost. Two models are developed based on the probability distribution of lead time demand following a normal distribution and distribution free respectively. From the results of numerical example, it can be shown that, the significant savings can be achieved through the reductions of order processing cost, safety factor and lost sales rate.**

***Index Terms*—Lead time, shortage, order processing cost, safety factor, EOQ.**

I. INTRODUCTION

Traditional inventory models assumed that lead time is a constant or random variable which is not a controllable factor. In fact, lead time could be shortened by paying an additional crashing cost, which could be expenditures on equipment improvement, information technology, order expedite, or special shipping and handling. By reducing lead time, one can decrease the stock out loss as improving the customer's satisfaction level. Thus, controllable lead time is a key to business achievement and it has attracted considerable research attention in present supply chain and inventory management system. As the same lead time, it is very important to improve the customer's satisfaction level.

Liao and Shyu [1] first devised a probabilistic inventory model in which lead time was the unique decision variable. Later many researchers developed lead time reduction inventory models under various crashing cost function and practical situations. Hsu and Lee [2] explored a single manufacturer multiple retailer integrated inventory system with the assumption of a non-increasing stair-step lead time crashing cost function. In contrast to existing inventory models, this paper considers two models that the crashing cost is an exponential function of lead time and the demand distribution follows a normal distribution and a distribution free, respectively.

Order processing cost reduction inventory research has received increasing attentions and hot topic in recent years. However, most of the existing inventory model assumed that order processing cost is fixed. In practice, order processing cost can be controlled and reduced through various efforts such as worker training, procedural changes and specialized equipment acquisition. Initially, Porteus [3] first investigated the impact of capital investment in reducing ordering cost on the classical economic order quantity model. Ouyang *et al.* [4] discussed ordering cost and lead time reductions in continuous review inventory systems with partial backorders. Later, Chang *et al.* [5] presented lead time and ordering cost reduction problem in the single vendor single buyer integrated inventory model. Jha and Shanker [6] presented an integrated production-inventory model where a vendor produces an item and supplies it to a set of buyers. Yi and Sarker [7] used controllable lead time in a buyer–vendor system. Giri and Roy [8] studied a two-echlon supply chain inventory system with a singal manufacturer and a single buyer, considering price dependent demand and variable lead time. Vijayashree and Uthayakumar [9] considered a single-vendor and a single-buyer integrated inventory model with ordering cost reduction dependent on lead time.They considered that lead time can be shortened at an extra crashing cost which depends on the lead time length to be reduced and the ordering lot size, as well buyer ordering cost can be reduced through further investment.

Recently, Annadurai and Uthayakumar [10] presented and analyzed a probabilistic inventory model under continuous review for the system with controllable lead time and optimal ordering cost caused by investment strategy subject to a service level constraint. Shahpouri *et al.* [11] assumed that shortages are either completely lost or completely backlogged in several inventory models. Montgomery *et al.* [12] is among the first who considered that a fraction of demand is back ordered and the remaining fraction is lost. Ouyang *et al.* [13] generalized Ben-Daya and Raouf's [14] model, where the backorder rate is fixed to a mixture of backorder and lost sales model. Ouyang and Chuang [15] investigated a mixture inventory model involving variable lead time and controllable backorder rate. They observe that many products of famous brands or fashionable goods such as certain brand gum shoes, hi-fi equipment, cosmetics and clothes may lead to a situation in which customers prefer their demands to be backordered when shortages occur. Harada *et al.* [16] presented an inventory model in which the order quantity, reorder point, and lead time are regarded as decision variables with controllable backorder rate.The stochastic inventory models analyzed in this paper involve two models that are continuous

Manuscript received May 19, 2018; revised August 9, 2018.

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doi: 10.18178/joebm.2018.6.3.557

review and periodic review in which the backorder rate is a random variable. Certainly, if the quantity of shortages is accumulated to a degree that exceeds the waiting patience of customers, some customers may refuse the backorder case. This phenomenon reveals that, as short ages occur, the longer the length of lead time is, the larger the amount of shortages is, the smaller the proportion of customers can wait and hence the smaller the backorder rate would be.

The above inventory models assumed that the quantity received is the same as the quantity ordered. But in real life circumstances, the quantity received may not match the quantity ordered due to various reasons such as damage or breakage during transportation, rejection during inspection, human errors in counting, transcribing, etc. In a recent paper, Kurdhi *et al.* [17] investigated continuous review inventory models involving service level constraint in which lead time, reorder point, ordering cost and order quantity are treated as decision variables and quantity received is uncertain. Priyan and Uthayakumar [18] explored a model in which the received quantity is uncertain. However, the safety factor, one key factor in inventory control policy, is not taken into account. Moreover, they assume the demand during lead time (DDLT) follow normal distribution. In most case, the distribution information of the lead time demand is very limited. In this paper, under the background of uncertain quantity received, we take safety factor into consideration and extend the DDLT from normal distribution to distribution free. However, the safety factor is not a decision variable in Priyan and Uthayakumar's [18] model.

In this paper, the lead time, safety factor, lost sales rate and order processing cost are decision variables. We minimize the expected annual total cost per unit time by simultaneously optimizing the order quantity, order processing cost, safety factor, backorder price discount, and lead time. We develop two inventory models with normal-distribution and distribution free DDLT under the background of uncertain quantity received. Furthermore, we provide numerical example to clarify the solution algorithm and to demonstrate the advantage of implementing the optimal inventory policy. Numerical results indicate that considerable cost savings could be realized through the optimal policy.

II. Notation and Assumptions

To model the problem, let first introduce the notation and assumptions of the model, part of which are adopt from Priyan [18].

A. Notations

Variable:

D average demand per year.

Y received quantity, a random variable.

α bias factor which is the expected amount received ÷ amount ordered, $0<\alpha<1$.

A_0 original ordering cost (before any investment is made).

h inventory holding cost per unit per year.

π fixed penalty cost per unit short.

π_0 marginal profit per unit.

β_0 original fraction of the shortage that will be lost.

X demand during lead time, a random variable.

$f(x)$ the probability distribution function (p.d.f) of X.

$E(\cdot)$ expected value.

x^+ maximum value of x and 0, i.e., $x^+ = max\{x,0\}$.

Decision variable:

k safety factor.

β fraction of the demand backordered during the stockout period, $0<\beta\le 1$, while the remaining fraction $(1-\beta)\equiv\tilde{\beta}$ is lost sales.

A ordering cost per order.

L length of lead time.

Q order quantity.

B. Assumptions

1. The quantity received is uncertain and depends on the quantity ordered.

2. The system uses a continuous review inventory policy and the order quantity Q is placed whenever the inventory level falls to the reorder point r. The reorder point r = expected DDLT $(DL)+$ the safety stock (S_s), and $S_s = kX$ (standard deviation of lead time demand), i.e. $r = DL + k\sigma\sqrt{L}$ where k is the safety factor and satisfies $Pr(X>r)=q$, q represents the allowable stock-out probability during lead time and is given.

3. The reorder point r doesn't affect the ordering policy of this inventory system since the total cost of the inventory system is independent of reorder point r.

4. The lead-time crashing cost per order, $R(L)$, is assumed to be an exponential function of L and is defined as

$$R(L)=\begin{cases}0 & L=L_0,\\ e^{C/L} & L_b\le L<L_0,\end{cases}$$

where C is a positive constant and L_0 and L_b represent the existing and the shortest lead time, respectively.

5. During the stockout period, a fraction β of the demand will be backordered, and the remaining fraction $(1-\beta)\equiv\tilde{\beta}$ will be lost. The ordering cost A and lost sales $\tilde{\beta}$ can be reduced by capital investment $I_0(A)$ and $I_1(\tilde{\beta})$, respectively.

III. Mathematical Models

Based on the above notations and assumptions, the expected annual total cost per cycle which is the sum of ordering cost, holding cost, stockout cost, and lead time crashing cost. Therefore, the retailer's total expected annual cost is given by

$$EAC(Q,r,L)=h\left[\frac{Q}{2}+r-DL+(1-\beta)E(X-r)^+\right] + A+\overline{\pi}E(X-r)^+ + R(L) \tag{1}$$

where $E(X-r)^+$ is the expected number of shortages per cycle and $\bar{\pi} = \pi + (1-\beta)\pi_0$.

This study considers the amount received is uncertain, that is, if a quantity Q is ordered each time, the quantity received will be Y which is a random variable with $E(Y \mid Q) = \alpha Q$.

Under the assumptions mentioned given before, the total cost per cycle with a variable lead time can be obtained as model (1) given that Y units are received is

$$E_Y(Y,r,L) = h\frac{Y}{D}\left[\frac{Y}{2} + r - DL + \tilde{\beta}E(X-r)^+\right] + A + (\pi + \tilde{\beta}\pi_0)E(X-r)^+ + R(L) \quad (2)$$

Therefore, the expected total cost per cycle with variable lead time when the amount received is uncertain is

$$E_X\left[E_Y(Y,r,L) \mid Q\right] = A + h\frac{\alpha Q}{D}\left[r - DL + \tilde{\beta}E(X-r)^+\right] + \frac{h}{2D}\left[\sigma_0^2 + (\sigma_1^2 + \alpha^2)Q^2\right] + (\pi + \tilde{\beta}\pi_0)E(X-r)^+ + R(L) \quad (3)$$

Moreover, the expected cycle time is

$$\frac{E(Y \mid Q)}{D} = \frac{\alpha Q}{D} \quad (4)$$

Hence, from (3) and (4) and (1) we obtain the expected annual total cost with variable lead time when the amount received is uncertain, denoted by $EAC(Q,r,L)$, is

$$E(Q,r,L) = \frac{AD}{\alpha Q} + h\left[r - DL + \tilde{\beta}E(X-r)^+\right] + \frac{h}{2\alpha Q}\left[\sigma_0^2 + (\sigma_1^2 + \alpha^2)Q^2\right] + \frac{D(\pi + \tilde{\beta}\pi_0)}{\alpha Q}E(X-r)^+ + \frac{R(L)D}{\alpha Q} \quad (5)$$

The relationship between ordering cost reduction and capital investment can be described by the logarithmic investment cost function. That is, ordering cost, A, and the capital investment in ordering cost reduction, I_0, can be stated as:

$$I_0(A) = c_1 ln(\frac{A_0}{A}) \text{ for } 0 < A \le A_0$$

Similarly, the relationship between lost sales, $\tilde{\beta}$, and capital investment in lost sales reduction, I_1, is described by

$$I_1(\tilde{\beta}) = c_2 ln(\frac{\beta_0}{\tilde{\beta}}) \text{ for } 0 < \tilde{\beta} \le \beta_0$$

where $1/c_1$ and $1/c_2$ are the fraction of the reduction in A and $\tilde{\beta}$ per dollar increase in investment, respectively. Therefore, the total investment in ordering cost and lost sales rate reduction is

$$I(A,\tilde{\beta}) = I_0(A) + I_1(\beta)$$

A. Normal Distribution Model

We assume that the lead time demand X has a normal p.d.f $f(x)$ with mean DL, standard deviation $\sigma\sqrt{L}$, and the reorder point $r = DL + k\sigma\sqrt{L}$ where k is the safety factor. Shortage occurs when $X > r$. Hence, the expected shortages at the end of the cycle time is given by

$$E(X-r)^+ = \int_r^\infty (x-r)f(x)dx = \sigma\sqrt{L}\Psi(k) \quad (6)$$

where $\Psi(k) = \phi(k) - k[1-\Phi(k)] > 0$. ϕ and Φ, are the standard normal probability density function and cumulative distribution function, respectively.

Hence, using (6) and $r = DL + k\sigma\sqrt{L}$, (5) can be reduced to

$$EAC(Q,k,L) = \frac{AD}{\alpha Q} + h[r - DL + \tilde{\beta}\sigma\sqrt{L}\Psi(k)] + \frac{h}{2\alpha Q}[\sigma_0^2 + (\sigma_1^2 + \alpha^2)Q^2] + \frac{(\pi + \tilde{\beta}\pi_0)}{\alpha Q}\sigma\sqrt{L}\Psi(k) + \frac{R(L)D}{\alpha Q} \quad (7)$$

Now our problem is to minimize the sum of the investment in ordering cost, lost sales rate reduction, and the inventory relevant costs as expressed in (7) by simultaneously optimizing $Q, A, \tilde{\beta}$ and L, constrained on $0 < A \le A_0$ and $0 < \tilde{\beta} \le \beta_0$. That is, the objective of our problem is to minimize the following expected annual total cost

$$\begin{aligned} EAC(Q,A,\tilde{\beta},k,L) &= \theta I(A,\tilde{\beta}) + EAC(Q,k,L) \\ &= \theta[I_0(A) + I_1(\tilde{\beta})] + \frac{AD}{\alpha Q} + h\sigma\sqrt{L} + [k + \tilde{\beta}\Psi(k)] \\ &+ \frac{h}{2\alpha Q}[\sigma_0^2 + (\sigma_1^2 + \alpha^2)Q^2] \\ &+ \frac{D(\pi + \tilde{\beta}\pi_0)}{\alpha Q}\sigma\sqrt{L}\Psi(k) + \frac{R(L)D}{\alpha Q} \end{aligned} \quad (8)$$

Subject to $0 < A \le A_0$ and $0 < \tilde{\beta} \le \beta_0$, where θ is the annual fractional cost of capital investment. Thus, our problem can be transformed to

$$\begin{aligned} minEAC(Q,A,\tilde{\beta},k,L) &= \theta\left\{c_1 ln(\frac{A_0}{A}) + c_2 ln\frac{\beta_0}{\tilde{\beta}}\right\} \\ &+ \frac{AD}{\alpha Q} + h\sigma\sqrt{L}[k + \tilde{\beta}\Psi(k)] + \frac{h}{2\alpha Q}[\sigma_0^2 + (\sigma_1^2 + \alpha^2)Q^2] \\ &+ \frac{D(\pi + \tilde{\beta}\pi_0)}{\alpha Q}\sigma\sqrt{L}\Psi(k) + \frac{R(L)D}{\alpha Q} \end{aligned} \quad (9)$$

where $\Psi(k) = \phi(k) - k[1-\Phi(k)] > 0$ and $L \in [L_b, L_0]$.

The problem formulated in the previous section appears as a constrained non-linear programming problem. In order to solve this kind of nonlinear problem, we follow the similar procedure of most of the literature dealing with nonlinear problem. That is, first we temporarily ignore the constraints of $0 < A \le A_0$ and $0 < \tilde{\beta} \le \beta_0$, then determine the optimum solutions of Q, k and $L \in [L_b, L_0], A \in (0, A_0]$ and $\tilde{\beta} \in (0, \beta_0]$ which minimizes expected annual total cost, $EAC(Q,A,\tilde{\beta},k,L)$. Initially, we can simplified $EAC(Q,A,\tilde{\beta},k,L)$ is a convex or concave function of

$L \in [L_b, L_0], A \in (0, A_0]$ and $\tilde{\beta} \in (0, \beta_0]$ for fixed Q and k by Lemma 1.

Lemma 1: For fixed $Q, A, \tilde{\beta}$, and k, $EAC(Q, A, \tilde{\beta}, k, L)$ is a convex or concave function of $L \in [L_b, L_0]$.

Proof: Taking the first and second partial derivatives of $EAC(Q, A, \tilde{\beta}, k, L)$ with respect to L, we can obtain

$$\frac{\partial EAC(Q,A,\tilde{\beta},k,L)}{\partial L} = \frac{\sigma h}{2\sqrt{L}}(k + \tilde{\beta}\Psi(k)) + \frac{\sigma\Psi(k)D}{2\alpha Q\sqrt{L}}(\pi + \tilde{\beta}\pi_0) - \frac{DCe^{C/L}}{\alpha QL^2}$$

and

$$\frac{\partial^2 EAC(Q,A,\tilde{\beta},k,L)}{\partial L^2} = -\frac{\sigma h}{4L^{3/2}}(k + \tilde{\beta}\Psi(k)) - \frac{\sigma\Psi(k)D}{2\alpha QL^{3/2}}(\pi + \tilde{\beta}\pi_0) + \frac{2DCe^{C/L}}{\alpha QL^2}\left(\frac{1}{L} + \frac{C}{2L^2}\right) \tag{10}$$

For fixed $Q, A, \tilde{\beta}$ and k, (10) can be rewritten as:

$$\frac{\partial^2 EAC(Q,A,\tilde{\beta},k,L)}{\partial L^2} = \frac{\sigma D}{4\alpha QL^{3/2}}(a - bQ) \tag{11}$$

where $a = \frac{8Ce^{C/L}}{\sigma\sqrt{L}}\left(\frac{1}{L} + \frac{C}{2L^2}\right)$ and $b = \frac{\alpha h}{D}(k + \tilde{\beta}\Psi(k)) + \Psi(k)(\pi + \tilde{\beta}\pi_0)$.

Based on (11), the sign of $\frac{\partial^2 EAC(Q,A,\tilde{\beta},k,L)}{\partial L^2}$ is determined by the value $(a - bQ)$, and it is obvious that $b > 0$. Since the trend of $EAC(Q, A, \tilde{\beta}, k, L)$ depends on the sign of $\frac{\partial^2 EAC(Q,A,\tilde{\beta},k,L)}{\partial L^2}$, we discuss the sign of $\frac{\partial^2 EAC(Q,A,\tilde{\beta},k,L)}{\partial L^2}$ in two cases:

Case 1. $(a - bQ) \geq 0$: $\frac{\partial^2 EAC(Q,A,\tilde{\beta},k,L)}{\partial L^2} \geq 0$. Hence, $EAC(Q, A, \tilde{\beta}, k, L)$ is a convex function of L on the interval $[L_b, L_{b-1}]$. In this case, the minimum expected annual total cost $EAC(Q, A, \tilde{\beta}, k, L)$ occurs on the interval $[L_b, L_{b-1}]$.

Case 2. $(a - bQ) < 0$: $\frac{\partial^2 EAC(Q,A,\tilde{\beta},k,L)}{\partial L^2} < 0$. Hence, $EAC(Q, A, \tilde{\beta}, k, L)$ is a concave function of L on the interval $[L_b, L_{b-1}]$. In this case, the minimum expected annual total cost, $EAC(Q, A, \tilde{\beta}, k, L)$, at the end point of the interval $[L_b, L_{b-1}]$

This completes the proof of Lemma 1.

Now, first we temporarily ignore the constrains $A \in (0, A_0]$ and $\tilde{\beta} \in (0, \beta_0]$ for fixed $[L_b, L_{b-1}]$, take the first partial derivatives of $EAC(Q, A, \tilde{\beta}, k, L)$ with respect to Q, k, A and $\tilde{\beta}$, we obtain

$$\frac{\partial EAC(Q,A,\tilde{\beta},k,L)}{\partial Q} = -\frac{AD}{\alpha Q^2} - \frac{h\sigma_0^2}{2\alpha Q^2} + \frac{h}{2\alpha}(\sigma_1^2 + \alpha^2) - \frac{D}{\alpha Q^2}(\pi + \tilde{\beta}\pi_0)\sigma\sqrt{L}\Psi(k) - \frac{R(L)D}{\alpha Q^2} \tag{12}$$

$$\frac{\partial EAC(Q,A,\tilde{\beta},k,L)}{\partial k} = h\sigma\sqrt{L} - h\tilde{\beta}\sigma\sqrt{L}\left[1 - \Phi(k)\right] - \frac{D(\pi + \tilde{\beta}\pi_0)}{\alpha Q}\sigma\sqrt{L}\left[1 - \Phi(k)\right] \tag{13}$$

$$\frac{\partial EAC(Q,A,\tilde{\beta},k,L)}{\partial A} = -\frac{c_1\theta}{A} + \frac{D}{\alpha Q} \tag{14}$$

$$\frac{\partial EAC(Q,A,\tilde{\beta},k,L)}{\partial \tilde{\beta}} = -\frac{c_2\theta}{\tilde{\beta}} + h\sigma\sqrt{L}\Psi(k) + \frac{D}{\alpha Q}(\pi_0\sigma\sqrt{L}\Psi(k)) \tag{15}$$

Then, by examining the second order sufficient conditions, it can be verified that $EAC(Q, A, \tilde{\beta}, k, L)$ is a convex function of Q and k for fixed $L \in [L_b, L_0], A \in (0, A_0]$ and $\tilde{\beta} \in (0, \beta_0]$.

On the other hand, for a given value of $L \in [L_b, L_0]$, by setting (12)-(15) equal to zero, we obtain

$$Q = \left\{\frac{2D\left[A + \frac{h\sigma_0^2}{2D} + (\pi + \tilde{\beta}\pi_0)\sigma\sqrt{L}\Psi(k) + R(L)\right]}{h(\sigma_1^2 + \alpha^2)}\right\}^{1/2} \tag{16}$$

$$\Phi(k) = 1 - \frac{h\alpha Q}{h\alpha Q\tilde{\beta} + D(\pi + \tilde{\beta}\pi_0)} \tag{17}$$

$$A = \frac{c_1\alpha\theta Q}{D} \tag{18}$$

$$\tilde{\beta} = \frac{c_2\alpha\theta Q}{\sigma\sqrt{L}\Psi(k)(h\alpha Q + D\pi_0)} \tag{19}$$

Theoretically, for fixed $L \in [L_b, L_0]$, by solving (16)-(19), we can obtain the values of $Q, A, \tilde{\beta}$ and k (denote these values by Q^*, A^*, β^* and $\Phi(k^*)$, respectively). The following proposition asserts that, for fixed $L \in [L_b, L_0], A \in (0, A_0]$ and $\tilde{\beta} \in (0, \beta_0]$, the point (Q^*, k^*) is the optimal solution. Hence that the expected total cost, $EAC(Q, A, \tilde{\beta}, k, L)$, has minimum value.

Based on the convexity and concavity behavior of objective function with respect to the decision variables the following algorithm is developed to find the optimal values for the order quantity, ordering cost, lost sales and lead time.

Algorithm:

Step 1. For each $L \in [L_b, L_0]$, repeat step (1.2) and (1.3) until no change occurs in the values of Q, k, A and $\tilde{\beta}$.

Denote the solution by $(Q', k', A', \tilde{\beta}')$.

Step 1.1. Start with $A_1 = A_0$ and $\beta_1 = \beta_0$.

Step 1.2. Substitute A_1 and β_1 into (16) and (17) evaluates Q_1 and k_1.

Step 1.3. Utilizing Q_1 and k_1 determines A_2 and β_2 from (18) and (19).

(i) If $A \le A_0'$ and $\tilde{\beta}' \le \beta_0$, then the solution found in step 2 is optimal for the given L. We denote the optimal solution by $(Q'', k'', A'', \tilde{\beta}'')$, i.e., if $(Q'', k'', A'', \tilde{\beta}'') = (Q', k', A', \tilde{\beta}')$, go to step 4.

(ii) If $A > A_0'$ and $\tilde{\beta}' \le \beta_0$, then for this given L, let $A' = A_0$ and utilize (16) and (17) (replace A by A_0), and (19) to determine the new Q'', k'' and $\tilde{\beta}''$ by a solution procedure similar to the one in step 1 (the result is denoted by $(Q''', \tilde{\beta}''')$. If $\tilde{\beta}''' \le \beta_0$, then the optimal solution is obtained, i.e., if $(Q', k', A', \tilde{\beta}') = (Q''', k''', A_0, \tilde{\beta}''')$, go to step 4; otherwise, go to step 3.

(iii) If $A' \le A_0$ and $\tilde{\beta}' > \beta_0$, then for this given L, let $\tilde{\beta}' > \beta_0$ and utilize (16) and (17) (replace $\tilde{\beta}'$ by β_0), and (19) to determine the new Q'', k'' and $\tilde{\beta}''$ by a solution procedure similar to the one in step 1 (the result is denoted by $(Q''', \tilde{\beta}''')$. If $A''' \le A_0$, then the optimal solution is obtained, i.e., if $(Q', k', A', \tilde{\beta}') = (Q''', k''', A''', \beta_0)$, go to step 4; otherwise, go to step 3.

(iv) If $A'' > A_0$ and $\tilde{\beta}'' > \beta_0$, go to step 3.

Step 3. For the given L, let $A' = A_0$ and $\tilde{\beta}' = \beta_0$, and utilize (16) and (17) (replace A by A_0 and $\tilde{\beta}'$ by β_0, to determine the corresponding optimal solution Q' and k' by a procedure similar to the one in step 1.

Step 4. Utilize (8) to calculate the corresponding expected annual total cost $EAC(Q, k, A, \tilde{\beta})$.

Step 5. Find $EAC(Q^*, k^*, A^*, \beta^*, L^*) = \text{Min}EAC(Q', k', A', \tilde{\beta}', L)$ for every $L \in [L_b, L_0]$, then $EAC(Q^*, k^*, A^*, \beta^*, L^*)$ is the minimum expected annual cost of proposed model, and $(Q^*, k^*, A^*, \beta^*, L^*)$ is the optimal solution. The reorder point $r^* = DL^* + k^*\sigma\sqrt{L^*}$.

B. Distribution Free Model

Now we consider the distribution free approach. Hence, a minimax distribution free procedure is applied to solving this problem.

$$MinMax_{F\in\Omega} EAC(Q, A, \tilde{\beta}, k, L) \tag{20}$$

Subject to $0 < A \le A_0$ and $0 < \tilde{\beta} \le \beta_0$.

The following proposition is used to approximate the value of $E(X-r)^+$.

Proposition 1: For any $F \in \Omega$,

$$E(X-r)^+ \le \frac{1}{2}\left\{\sqrt{\sigma^2 L + (r - DL)^2} - (r - DL)\right\} \tag{21}$$

Substituting $r = DL + k\sigma\sqrt{L}$ into (21), we obtain the following inequality

$$E(X-r)^+ \le \frac{1}{2}\sigma\sqrt{L}(\sqrt{1+k^2} - k) \tag{22}$$

Now using (3) and inequality (22), (20) reduces to

$$\begin{aligned} &EAC^W(Q, A, \tilde{\beta}, k, L) = \theta(A, \tilde{\beta}) + EAC^W(Q, k, L) \\ &= \theta[I_0(A) + I_1(\tilde{\beta})] + \frac{AD}{\alpha Q} + h\sigma\sqrt{L}[k + \frac{\tilde{\beta}}{2}(\sqrt{1+k^2} - k)] \\ &+ \frac{h}{2\alpha Q}\left[\sigma_0^2 + (\sigma_1^2 + \alpha^2)Q^2\right] + \frac{D(\pi + \tilde{\beta}\pi_0)\sigma\sqrt{L}}{2\alpha Q}(\sqrt{1+k^2} - k) \\ &+ \frac{R(L)D}{\alpha Q} \end{aligned} \tag{23}$$

Subject to $0 < A \le A_0$ and $0 < \tilde{\beta} \le \beta_0$.

We first ignore two constraints and obtain

$$\begin{aligned} &\frac{\partial EAC^W(Q, A, \tilde{\beta}, k, L)}{\partial L} = \frac{h\sigma}{2\sqrt{L}}[k + \frac{\tilde{\beta}}{2}(\sqrt{1+k^2} - k)] \\ &+ \frac{D\sigma(\pi + \tilde{\beta}\pi_0)}{4\alpha Q\sqrt{L}}(\sqrt{1+k^2} - k) - \frac{DCe^{C/L}}{\alpha QL^2} \end{aligned} \tag{24}$$

$$\begin{aligned} &\frac{\partial^2 EAC^W(Q, A, \tilde{\beta}, k, L)}{\partial L^2} = -\frac{h\sigma}{L^{3/2}}[k + \frac{\tilde{\beta}}{2}(\sqrt{1+k^2} - k)] \\ &- \frac{D\sigma(\pi + \tilde{\beta}\pi_0)}{4\alpha QL^{3/2}}(\sqrt{1+k^2} - k) + \frac{2DCe^{C/L}}{\alpha QL^3}(1 + \frac{C}{2L}) \end{aligned} \tag{25}$$

On the other hand, we temporarily ignore the constraints $A \in (0, A_0]$ and $\beta^* \in (0, \beta_0]$, then for a given value of $L \in [L_b, L_0]$, we obtain

$$Q = \left\{\frac{2D[A + \frac{h\sigma_0^2}{2D} + (\pi + \tilde{\beta}\pi_0)(\sqrt{1+k^2} - k)\sigma\sqrt{L} + R(L)]}{h(\sigma_1^2 + \alpha^2)}\right\}^{1/2} \tag{26}$$

$$\frac{k}{\sqrt{1+k^2}} = 1 - \frac{h\alpha Q}{h\alpha Q\tilde{\beta} + D(\pi + \tilde{\beta}\pi_0)} \tag{27}$$

$$A = \frac{c_1\alpha\theta Q}{D} \tag{28}$$

$$\tilde{\beta} = \frac{2c_2\alpha\theta Q}{\sigma\sqrt{L}(\sqrt{1+k^2} - k)(h\alpha Q + D\pi_0)} \tag{29}$$

For fixed $Q, A, \tilde{\beta}$ and k, (25) can be rewritten as:

$$\frac{\partial^2 EAC(Q, A, \tilde{\beta}, k, L)}{\partial L^2} = \frac{\sigma D}{4\alpha QL^{3/2}}(a - bQ) \tag{30}$$

where $a = \frac{8Ce^{C/L}}{\sigma\sqrt{L}}(\frac{1}{L} + \frac{C}{2L^2})$ and $b = (\frac{\alpha h}{D} + \pi + \tilde{\beta}\pi_0)(\sqrt{1+k^2} - k)$.

Based on (30), the sign of $\frac{\partial^2 EAC^W(Q,A,\tilde{\beta},k,L)}{\partial L^2}$ is determined by the value $(a-bQ)$, and it is obvious that $b>0$. Since the trend of $EAC^W(Q,A,\tilde{\beta},k,L)$ depends on the sign of $\frac{\partial^2 EAC^W(Q,A,\tilde{\beta},k,L)}{\partial L^2}$, similar normal distribution we discuss the sign of $\frac{\partial^2 EAC^W(Q,A,\tilde{\beta},k,L)}{\partial L^2}$ in two cases:

Case 3. $(a-bQ)\geq 0$: $\frac{\partial^2 EAC^W(Q,A,\tilde{\beta},k,L)}{\partial L^2}\geq 0$. Hence, $EAC^W(Q,A,\tilde{\beta},k,L)$ is a convex function of L on the interval $[L_b,L_{b-1}]$. In this case, the minimum expected annual total cost, $EAC^W(Q,A,\tilde{\beta},k,L)$ on the interval $[L_b,L_{b-1}]$.

Case 4. $(a-bQ)<0$: $\frac{\partial^2 EAC^W(Q,A,\tilde{\beta},k,L)}{\partial L^2}<0$. Hence, $EAC^W(Q,A,\tilde{\beta},k,L)$ is a concave function of L on the interval $[L_b,L_{b-1}]$. In this case, the minimum expected annual total cost, $EAC(Q,A,\tilde{\beta},k,L)$ at the end point of the interval $[L_b,L_{b-1}]$.

IV. Numerical Analysis

To illustrate the above issues, the values of the following parameters which are almost similar to those use in : $D=600$ unit/year, $h=\$20$ /unit/year, $\pi=\$20$ /unit/year, $\pi_0=\$150$ /unit/year $\sigma=7$ /unit/week, $\sigma_0^2=100$, $\sigma_1^2=0.1$ and $\alpha=0.9$, $\beta_0=0.3$, $C=30$, $L_0=10$ days and $L_b=6$ days. Besides, we consider a situation where the initial ordering cost $A_0=200$ /order, initial lost sales rate $\beta_0=(1-\beta_0)=0.7$, $\theta=0.1$, $c_1=5800$ and $c_2=0.2$.

A. Effects of Parameter on Optimal Solution

The demand D, holding cost h, investment cost c_1 and c_2 sensitivity analysis are performed in order to understand how various D, h, c_1, and c_2 affect the optimal solution of the model. The example 1 and example 2 use a normal distribution and distribution free, respectively. In addition, we compare our numerical with Priyan and Uthayakumar's [18] model to illustrate the effect of safety factor as a decision variable. The summarization of the comparison is shown in Table I-Table IV.

In this world no one can accurately predict his customer demand in advance. Therefore, the assumptions of uncertain demand is might appropriate for all industries in this world. Additionally, when the demand is uncertain, reorder point becomes an important issue and its control leads to several benefits. Shorter lead time reduces the safety stock and the loss caused by stock-out, improves customer service level and increases the competitive advantage of business. Here we present the managerial implications of proposed model based on the numerical results and effect of model parameters.

Table I shows that the lot size Q, ordering cost A and expected total cost $EAC^W(Q^*,A^*,\ \beta^*,L^*)$ decreases when the safety factor k increase. Introducing the variable safety factor, we obtained a better result than Priyan *et al.*'s [18] when the safety factor was determined for $k=0.25$ (see Table I-Table IV).

1. Table I shows that when demand D increases, the expected total cost $EAC^W(Q^*,A^*,\beta^*,L^*)$ increase.
2. In Table II, it is interesting to note that when holding cost h increases, the optimal lot size Q, ordering cost A, lost sales $\tilde{\beta}$ decreases and expected annual total cost $EAC(Q^*,A^*,\beta^*,L^*)$ increases without affecting the lead time L. This result is expected because higher holding cost may amplify total cost in the real marketing.
3. Table III shows that when the investment cost function c_1 increase, the optimal lot size Q, ordering cost A, lost sales $\tilde{\beta}$ and expected annual total cost $EAC(Q^*,A^*,\beta^*,L^*)$ increases without affecting the lead time L.
4. It is interesting to note that expected total cost $EAC(Q^*,A^*,\beta^*,L^*)$ increases when the c_2 increases without affecting the lead time (see Table IV).

B. Evaluation of Expected Value of Additional Information $(EVAI)$

Now, we compare the results of distribution free model with the normal distribution model $EVAI$. If we utilize the solution $(Q^*,k^*,A^*,\beta^*,L^*)$ obtained by the distribution free approach instead of utilizing (Q^*,A^*,β^*,L^*) from the normal distribution case, then the added cost will be $EAC^W(Q^*,k^*,A^*,\beta^*,L^*)-EAC(Q^*,A^*,\beta^*,L^*)$. This amount is the expected value of additional information that the buyer would be willing to pay for the information regarding the nature of lead time demand distribution (see Table V).

TABLE I: Effects of Demand D on Optimal Solution of Example 1

Reorder point as a decision variable							Fixed $k=0.25$ in [18]		Saving(%)
D	L	Q^*	A^*	$\beta^*(10^{-5})$	k^*	$EAC(\cdot)$	$(Q^*,A^*,\beta^*(10^{-4}))$	$EAC(\cdot)$	
400	8	100	130.2	1.1330	0.757	2618.2	(152,200,1.92)	3186	17.82
600	8	106	91.8	1.2568	1.006	3015.3	(179,155,1.52)	3859	21.86
800	8	112	72.7	1.3336	1.1490	3319.2	(201,131,1.29)	4408	24.70
1200	8	123	53.2	1.4299	1.3302	3785.4	(238,104,1.03)	5303	28.62

TABLE II: EFFECTS OF HOLDING COST h ON OPTIMAL SOLUTION OF EXAMPLE 1

Reorder point as a decision variable							Fixed $k=0.25$ in [18]		Saving(%)
h	L	Q^*	A^*	$\beta^*(10^{-5})$	k^*	$EAC(\cdot)$	$(Q^*, A^*, \beta^*(10^{-4}))$	$EAC(\cdot)$	
10	8	169	146.3	2.6632	1.1449	2121.2	(264,200,1.45)	2719	21.99
15	8	127	110.4	1.7219	1.0683	2608.0	(215,198,1.84)	3344	22.01
20	8	106	91.8	1.2568	1.0016	3015.3	(179,155,1.52)	3859	21.86
25	8	93	80.4	0.9807	0.9416	3376.4	(145,126,1.23)	4317	21.79
30	8	84	72.6	0.7985	0.8863	3706.7	(126,110,1.07)	4731	21.65

TABLE III: EFFECTS OF INVESTMENT COST FUNCTION c_1 ON OPTIMAL SOLUTION OF EXAMPLE 1

Reorder point as a decision variable							Fixed $k=0.25$ in [18]		Saving(%).
c_1	L	Q^*	A^*	$\beta^*(10^{-5})$	k^*	$EAC(\cdot)$	$(Q^*, A^*, \beta^*(10^{-4}))$	$EAC(\cdot)$	
4500	8	96	64.2	1.2913	1.0681	2887.6	(171,115,1.45)	3808	24.17
5800	8	106	91.8	1.2568	1.0016	3015.3	(179,155,1.52)	3859	21.86
6500	8	112	108.5	1.2387	0.9664	3066.1	(182,177,1.55)	3872	20.81
7200	8	118	126.6	1.2209	0.9317	3108.8	(187,200,1.60)	3875	19.77

TABLE IV: EFFECTS OF INVESTMENT COST FUNCTION c_2 ON OPTIMAL SOLUTION OF EXAMPLE 1

Reorder point as a decision variable							Fixed $k=0.25$ in [18]		Saving(%).
c_2	L	Q^*	A^*	$\beta^*(10^{-5})$	k^*	$EAC(\cdot)$	$(Q^*, A^*, \beta^*(10^{-4}))$	$EAC(\cdot)$	
0.1	8	106	91.8	0.62835	1.0015	3015.2	(179,156,3.05)	3863	21.95
0.2	8	106	91.8	1.2568	1.0016	3015.3	(179,155,1.52)	3859	21.86
0.3	8	106	91.8	1.8853	1.0016	3015.4	(178,154,0.91)	3857	21.80
0.5	8	106	91.8	3.1425	1.0017	3015.6	(177,152,0.25)	3854	21.75

TABLE V: THE OPTIMAL SOLUTION OF EXAMPLE 2

D	L	Q^*	A^*	$\beta^*(10^{-5})$	k^*	$EAC(\cdot)$	$EVAI$
400	8	128	167.6	0.912	1.0111	2819.1	200.9
600	8	141	123.1	0.807	1.2787	3328.5	313.2
800	8	152	99.5	1.4751	1.4791	3721.7	402.5
1200	8	170	74.3	1.2970	1.2970	4329.6	544.2

V. CONCLUSION

In the real world, the retailer can't accurate prediction customer's demand, received quantity, etc., in advance. Therefore, the assumptions of uncertain demand and received quantity are might appropriate for all industries in this world. Additionally, the lead time, order received rate, safety factor and backorder play important roles in inventory control policy. A continuous review inventory model is developed to investigate the effects of the optimal solution, in which capital investment strategies in order processing cost and lost sales rate reduction are adopted in the uncertain demand. By analyzing the expected annual total cost, we develop algorithm to determine the optimal order quantity, ordering cost, backorder price discount, safety factor, lost sales rate and lead time simultaneously. The results of the numerical examples indicate that if we make decisions with the capital investment in reducing order processing cost, lost sales rate and offering backorder price discount to customers, it will help to lower the system cost. Therefore we can obtain a significant amount of savings to increase the competitive edge in business. Future research may consider multi items and permissible delay in payments and fuzzy demand in this model. It would be interesting to consider the procurement lead time as a random variable and discuss the effects in reducing lead time variability. Another possible extension of this work may be conducted by considering lead time dependent partial backlogging in this inventory model.

ACKNOWLEDGMENT

This research is supported by the general program of National Natural Science of China (Grant No. 71172195).

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