Modelling of the Economic Production Quantity under Two Stages of a Product Life Cycle

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***Abstract*—Considering a finite time horizon across multiple stages of a product life cycle, this study presents a production inventory model with multiple demand rates under a non-periodic policy. In real, the demand of the product life cycle is a non-linear function but this work assumes it as four-segment linear or constant approximations. In addition, a multiple-segment to be combined with linear or constant functions can be approximated a nonlinear function. This study does not focus on this, but it provides a simple algorithm to easily deal with this proposed inventory problems. Although our approach is a heuristics, our algorithm is better than the sum of two independent single-piece linear or constant models on the total cost's performance. A general procedure for solving three demand types, including a piecewise linear, linear-constant and constant-linear trend in demand, is provided.**

***Index Terms*—Economic production quantity (EPQ), linear trend in demand, multiple stages, product life cycle (PLC).**

I. Introduction

Product life cycle is the progression of an item through the four stages of its time on the market. The four life cycle stages are: Introduction, Growth, Maturity and Decline. Due to advanced technology and intense competition, the short product life cycle that consists a slow increasing, rapidly increasing, steady and decreasing trend in demand, becomes one of common characteristics for high technology products, which an entire life cycle time may be ranging from nine to eighteen months [1]. In today's time based competition, the collaborative planning, forecasting and replenishment (CPFR), that aims to enhance supply chain integration between a buyer and vendor, has become essential. Under the CPFR, the vendor would request the buyer to build up a medium- to long-term proposal replenishment schedule for his collaborative forecast and inventory management. Consequently, the planner of the vendor has to map out the interval and demand of each stage and set up an intermediate-range production plan based to the buyer's forecast. This planning horizon is maybe across over two stages on product life cycle.

For the phenomenon of product life cycle, most previous studies have assumed that the demand pattern is one linear function on production or inventory problems. However, these researches consider only one linear function over the planning horizon that is not capable of handling the above situation. One primitive solution following previous one linear function algorithm is to sum each separated stage with the single-piece linear model; however, it is neither practical nor effective. To acquire a better solution and release a strict assumption that is no inventory to be held at the end of each stage, this study proposes an algorithm for the production policy when the trend of demand is a piecewise linear function.

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II. Literature Review

To consider an increasing demand on the product life cycle or bloom season, Resh [2] was the first author to introduce an inventory model with time-proportional demand. Donaldson [3] proposed an analytic approach for replenishment problem with a linear (increasing) trend in demand. Based on the above contribution, Henery [4] was the first to put forward a recursive procedure for determining the optimal replenishment schedule under the condition of fixed replenishment lots. Hariga's iterative algorithm [5] replied upon replenishment period that specified the proportion of two adjacent periods for both increasing and decreasing demand. Hill [6] was the first to investigate how batching policies were determined under a linearly increasing demand under a finite production rate. Omar and Smith [7] modified Hill's model [6] for linearly increasing demand in order to derive the optimal solution for integrating the batch size of raw materials and production lots in a manufacturing system. Rau and Ouyang [8] presented a general and simple algorithm to obtain an optimal solution for three inventory models on replenishment-batching, production-batching and integrated both replenishment and production batching policy in a manufacturing system. Actually, a piecewise linear function can described as an approximation of nonlinear function. To break the demand pattern up into segments, Hill [9] first introduced a general, time-varying, continuous, deterministic demand pattern through a complete product life cycle. Rau and Ouyang [10] investigated how to model an economic order quantity with a piecewise linear trend in demand. Therefore, none of the above researches can solve the proposed EPQ problem. In this paper, we extend this algorithm Rau and Ouyang [10] to this piecewise case and provide a simple heuristics to easily solve economic production quantity problem. In subsequent sections, first, we briefly review the classical model with a linear trend in demand. Then, we extend this algorithm to a two stages of the product life cycle model. Finally, we conduct the procedures to derive the solution of these inventory models.

III. Assumptions and Notation

To develop our proposed model, the following assumptions and notation are used.

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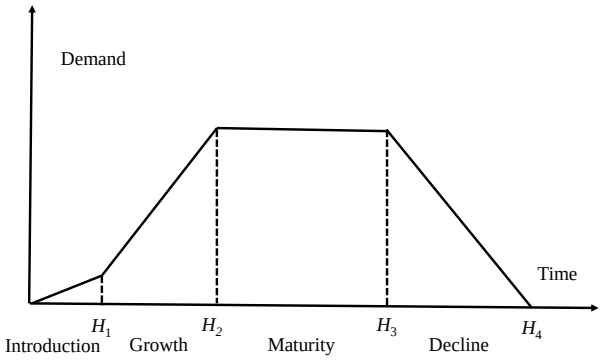

Fig. 1. Demand pattern for this proposed product life cycle.

A. Assumptions

According to characteristics of Product life cycle stages, the demand starts to increase slowly and gradually in the introduction stage. Next, the demand increase rapidly during the growth stage. Then, the demand becomes to steady over the maturity stage. Finally, the demand ends to decrease fast under the decline stage. In real, the demand of the product life cycle is a non-linear function but we treat it as four-segment linear or constant approximations in here. The demand pattern of this proposed product life cycle is shown in Fig. 1. Except the above assumption, we still have some relative inventory restrictions in our study as follows:

- A single item is considered.
- A finite time horizon is considered.
- The demand is a continuous piecewise function under a planning time horizon that consists of two pieces, each of which is a constant, linearly increasing or decreasing function.
- The ending point of the first stage is the starting point of the second stage and time is continuous.
- Unit production cost is deterministic and fixed.
- Production capacity is always greater than demand.
- Lead-time or shortage is not considered.
- Demand is always greater than zero at the end of the time horizon.
- No stock is held at the beginning and the end of the time horizon.

B. Notation

n	number of production cycles.
n_0	number of production cycles obtained by EOQ model.
H_i	the length of the first stage. i = 1,2,3,4,
H	finite time horizon.
W	total cost, including set up and holding cost.
$W^*(n)$	optimal total cost for n production cycles.
c_1	set up cost per order.
c_2	holding cost per unit per year.
$f(t)$	demand function during the time horizon.
$f_1(t)$	demand function during the first stage.
$f_2(t)$	demand function during the second stage.
a_1	demand at t = 0 (the starting point of the first stage)
b_1	rate of demand change per unit of time during stage one.
b_2	rate of demand change per unit of time for at stage two.
t_i	Terminating time of ith production cycles time or starting time of i+1th production cycles period i = 0, 1,…, n-1.
T_i	interval of the ith production cycle, i = 1,2,…,n, where $T_i = t_i - t_{i-1}$.
T_k	length of the kth production cycles period across stage one and stage two.
$f_m(t_i)$	demand at midpoint of the ith production cycle and $f_m(t_i) = f(\frac{t_i + t_{i-1}}{2})$

IV. The Mathematical Development

Under a finite time horizon, the economic production quantity model for a linear or constant trend in demand is an unconstrained nonlinear problem, where we have to determine the production schedule t_i and number of production cycles n over a fixed planning horizon H. A specified production cycles n is assumed in the following mathematical development.

A. Linear Demand

The introduction, growth or decline stage is a linear trend in demand. Therefore, the total relevant cost, including production set up cost and holding cost, is [8]:

$$W(n) = nc_1 + c_2 \sum_{i=0}^{n-1} \left[\int_{t_i}^{t_{i+1}} \int_{t}^{t_{i+1}} f(u)dudt - \left(\int_{t_i}^{t_{i+1}} f(t)dt \right)^2 / 2P \right] \quad i = 1, 2, \ldots n-1. \tag{1}$$

From equation (1), it is easy to understand that the production schedule t_i and the number of production lots n have to be decided in the planning horizon H. First, we solve the best production schedule. Under a fixed n, the derivative of equation (1) with respect to t_i, i =1,2….n-1, is

$$\frac{\partial W(n)}{\partial t_i} = c_2 \left[(t_i - t_{i-1}) f(t_i) - \int_{t_i}^{t_{i+1}} f(t)dt + f(t_i) \left(\int_{t_i}^{t_{i+1}} f(t)dt - \int_{t_{i-1}}^{t_i} f(t)dt \right) / P \right]. \tag{2}$$

B. Constant Demand

When demand rate is equal to zero in Eq. (1). The total relevant cost of maturity stage, including production set up cost and holding cost, is given by

$$W(n) = nc_1 + c_2 \sum_{i=0}^{n-1} \left[(P - f(t)) \left(\int_{t_i}^{t_{i+1}} f(t)dt \right) (t_{i+1} - t_i) / 2P \right] \quad i = 1, 2, \ldots n-1. \tag{3}$$

Similarly, the derivative of equation (3) with respect to t_i, for i =1,2….n-1, derives

$$\frac{\partial W(n)}{\partial t_i} = c_2 * (p - f(t)) * \left(\int_{t_{i-1}}^{t_i} f(t)dt - \int_{t_i}^{t_{i+1}} f(t)dt \right) / P. \tag{4}$$

C. Piecewise Demand

Under the collaborative planning, forecasting and

replenishment (CPFR), we suppose that the vendor have a planning time horizon H. The demand function during the first stage is $f_1(t) = a_1 + b_1 t$ with a known period H_1 and the demand function during the second stage (from H_1 to H) is $f_2(t) = a_2 + b_2 t$ and $a_2 = a_1 + (b_1 - b_2)H_1$ from the buyer's forecast. The ending point of the first stage is the starting point of the second stage. Hence, the demand function under the planning horizon can be expressed as follows:

$$f(t) = \begin{cases} a_1 + b_1 t & t < H_1 \\ a_2 + b_2 t & else \end{cases}. \tag{5}$$

The demand function $f(t)$ is a continuous function of time t but not differentiable only at H_1. So, it has to be separately processed by the following two cases.

Case one: If the kth production cycle (k is less than n) precisely occurs at H_1, this model can be handled with two single-segment linear models. The solution for this case refers to [11]

Case two: Under a fixed number of production cycles n over the time horizon, suppose the kth production period is across segments one and two and there are two linear demand or constant functions $f_1(t)$ and $f_2(t)$ at the kth production period, Referring to the above single-piece linear model, the total time-weighted stockholding, including $t \le H_1$, $H_1 \le t \le t_k$ and $t_k \le t \le H$ can be expressed

$$\sum_{i=0}^{k-2}\left[\int_{t_i}^{t_{i+1}}\int_{t}^{t_{i+1}} f_1(u)dudt - \left(\int_{t_i}^{t_{i+1}} f_1(t)dt\right)^2 / 2P\right]$$
when $t \le H_1$

$$\int_{t_{k-1}}^{H_1}\int_{t}^{H_1} f_1(u)dudt + (H_1 - t_{k-1})\int_{H_1}^{t_k} f_1(t)dt$$
$$+\int_{H_1}^{t_k}\int_{t}^{t_k} f_2(u)dudt - \left(\int_{t_i}^{H_1} f_1(t)dt + \int_{H_1}^{t_k} f_2(t)dt\right)^2 / 2P \tag{6}$$
when $t_i < H_1$ and $t_{i+1} > H_1$

$$\sum_{i=k}^{n-1}\left[\int_{t_i}^{t_{i+1}}\int_{t}^{t_{i+1}} f_2(u)dudt - \left(\int_{t_i}^{t_{i+1}} f_2(t)dt\right)^2 / 2P\right]$$
when $t_k \le t \le H$

Thus, the total cost, including the set up cost and holding cost, is given by

$$\begin{aligned} W = {} & nc_1 + c_2\sum_{i=0}^{k-2}\left[\int_{t_i}^{t_{i+1}}\int_{t}^{t_{i+1}} f_1(u)dudt - \left(\int_{t_i}^{t_{i+1}} f_1(t)dt\right)^2 / 2P\right] \\ & + c_2\left[\int_{t_{k-1}}^{H_1}\int_{t}^{H_1} f_1(u)dudt + (H_1 - t_{k-1})\int_{H_1}^{t_k} f_1(t)dt \right. \\ & \left. + \int_{H_1}^{t_k}\int_{t}^{t_k} f_2(u)dudt - \left(\int_{t_i}^{H_1} f_1(t)dt + \int_{H_1}^{t_k} f_2(t)dt\right)^2 / 2P\right] \\ & + c_2\sum_{i=k}^{n-1}\left[\int_{t_i}^{t_{i+1}}\int_{t}^{t_{i+1}} f_2(u)dudt - \left(\int_{t_i}^{t_{i+1}} f_2(t)dt\right)^2 / 2P\right] \end{aligned} \tag{7}$$

Here, our heuristic method conjectures that W is differentiable and it can be found its minimum at $\partial W / \partial t_i = 0$. Differentiate (7) with respect to t_i and we have

$$\begin{aligned} \frac{\partial W(n)}{\partial t_i} = {} & c_2\left[(t_i - t_{i-1}) f_1(t_i) - \int_{t_i}^{t_{i+1}} f_1(t)dt \right. \\ & \left. + f(t_i)\left(\int_{t_i}^{t_{i+1}} f_1(t)dt - \int_{t_{i-1}}^{t_i} f_1(t)dt\right)/P\right]. \qquad i = 1,..,k-2 \end{aligned},$$

$$\begin{aligned} \frac{\partial W(n)}{\partial t_i} = {} & c_2\left[(t_i - t_{i-1}) f(t_i) - \int_{t_i}^{t_{i+1}} f(t)dt \right. \\ & \left. + f(t_i)\left(\int_{t_i}^{t_{i+1}} f(t)dt - \int_{t_{i-1}}^{t_i} f(t)dt\right)/P\right]. \qquad i = k-1 \end{aligned}$$

$$\begin{aligned} \frac{\partial W(n)}{\partial t_i} = {} & c_2\left[(t_i - t_{i-1}) f_2(t_i) - \int_{t_i}^{t_{i+1}} f_2(t)dt \right. \\ & \left. + f(t_i)\left(\int_{t_i}^{t_{i+1}} f_2(t)dt - \int_{t_{i-1}}^{t_i} f_2(t)dt\right)/P\right]. \qquad i = k,..,n-1 \end{aligned} \tag{8}$$

We summarize (8) and derive the production schedule in the following equation

$$T_{i+1} = \begin{cases} \prod_{j=2}^{i} \dfrac{f_1(t_j)(P - f_{1m}(t_j))}{f_{1m}(t_{j+1})(P - f_1(t_j))} T_1 \\ t_{i+1} < H_1 \\ \dfrac{\left(f_1((\frac{t_{k-1}+H_1}{2})(H_1 - t_{k-1}) + f_2((\frac{t_k + H_1}{2})(t_k - H_1)\right)}{f_1((t_i)T_{k-1}} \\ t_i < H_1 \quad and \quad t_{i+1} > H_1 \\ \prod_{j=k}^{i} \dfrac{f_2(t_j)(P - f_{2m}(t_j))}{f_{2m}(t_{j+1})(P - f_2(t_j))} T_k \\ t_{i+1} > H \end{cases}$$

$$\text{for} \quad f(t_j) = f\left(\sum_{k=1}^{i} T_k\right) \quad \text{and} \quad \sum_{i=1}^{n} T_i = H \tag{9}$$

V. Solution procedures

Based on the above mathematical model development, the optimal solution of the inventory production policy for the single-piece linear model and the heuristic solution for the piecewise linear model can be conducted in a general procedure as follows:

Step 1. Calculate n0 and let $n = n_0$, where n_0 is an EOQ-based estimator, as shown in the following equation in order to elevate the searching performance.

$$n_0 = \left\lceil \sqrt{\frac{c_2 H_1(a_1 + b_1 H_1)}{2c_1}} \right\rceil + \left\lceil \sqrt{\frac{c_2(H - H_1)[(a_1 + b_1 H_1) + b_2(H - H_1)]}{2c_1}} \right\rceil \tag{10}$$

Step 2. $t_n = H$ and $T_1 = H/n$.

Step 3. $T_1 = t_n/n$.

Step 4. Solve T_i with equation (9) by an iterative procedure until a satisfied EPS occurs. Compute the total cost $W^*(n)$ from equation (7).

Step 5. Compute the total cost $W^*(n\text{-}1)$ and $W^*(n+1)$ by performing Steps 3 and 4 with respect to n-1 and n+1.

Step 6. Select Case Min{$W^*(n\text{-}1)$, $W^*(n)$, $W^*(n+1)$}

Case W*(n-1)

$n = n$-1

If n = 1, then Go to Step 7, else Go to Step 2.

Case W*(n)

Go to Step 7.

Case W*(n+1)

$n = n$+1 : Go to Step 2.

End Select

Step 7. The number of production cycle is n*. The production schedule is T_i, i = 0,1,…,n-1 and the total cost is W*(n).

In the above procedure, initiate an approximate value for the number of production lots, n_0, from equation (9) first, and then with a trial value for starting point T_1 (or t_1) in Steps 2 and 3, solve T_2, T_3,…..,T_{n-1}, T_n in Step 4 and get. New t_n is ether larger or smaller than H. After that, we adjust a new T_1 in Step 3. Execute Step 4 again and tn will be closer to H. After a few iterations, t_i can be determined if a pre-defined convergence criterion (EPS) is reached, which must be specified in the Step 4, then the best schedule is derived. The criterion is shown in equation (11), which states that the difference between the k+1th and the kth iterations of t_i's convergence is less than a specified value.

$$\text{Max}\left|\frac{t_i^{k+1}-t_i^k}{H}\right| \le \text{EPS}, \quad i = 1,2,....,n-1. \tag{11}$$

To perform a one-dimension search, we can obtain the optimal number of productions, n*, when W*(n) is the minimum.

VI. Conclusion

This study extended the algorithm for the classical production policy with a single-segment linear function in demand to provide a heuristic solution for the inventory replenishment model with a two-segment linear function in demand under two continuous stages of the product life cycle. Actually, the two-segment piecewise linear demand model can be easily extended into multi-segment models. In addition, these multi-segment models can also be approximated as a nonlinear function, such as trigonometric, exponential or cubic equation. In terms of the future research, people could consider other possible ways of solving this problem, more complex inventory policies, or relative issues in supply chain environment with multiple stages of the product life cycle. However, we believe that the approach of our paper serves as a good starting point for all these future researches.

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